1. Hochschild homology

Given an associative algebra A over a commutative ring R, the Hochschild homology of A with coefficients in a A-bimodule is the homotopy of the simplicial abelian group

$$\mathrm{Bar}^R(A) \underset{A\otimes A^{\mathrm{op}}}{\otimes} M = \mathrm{Bar}^R_{\mathrm{cyc}}(A, M)$$

or, equivalently, as the homotopy of its realization. In that expression, $\mathrm{Bar}^R(A)$ stands for the bar resolution of A over R. We will denote Hochschild homology by $HH^R_*(A, M)$.

2. Topological Hochschild homology

Topological Hochschild homology is defined analogously as:

$$THH^R(A, M) = |\,\mathrm{Bar}^R_{\mathrm{cyc}}(A, M)|$$

for an associative (cofibrant) R-algebra A and a A-bimodule M. R is here a (cofibrant) commutative S-algebra. We'll abbreviate $THH^R(A, A)$ as $THH^R(A)$. [cofibrant could be simply replaced by units being cofibrations]

This immediately gives us the following relation:

$$THH^{HR}_*(HA, HM) = HH^R_*(A, M)$$

for a commutative ring R, a flat R-algebra A and a A-bimodule M. This follows because in this case $HA \wedge_{HR} HN \simeq H(A \otimes_R N)$ for any N.

Also, the spectral sequence for a simplicial spectrum gives us the Bökstedt spectral sequence

$$HH^{E_*R}_*(E_*A, E_*M) \Longrightarrow E_*(THH^R(A, M))$$

when $E_*(A)$ is $E_*(R)$-flat.

In what follows, the references to R will often be removed.

3. The case of commutative algebras

For a commutative R-algebra A, we have

$$THH(A) \simeq S^1 \otimes A \tag{1}$$

where "$\otimes$" in the right-hand side indicates the enrichment of commutative R-algebras over sSet. A proof of this statement comes from noticing that the enrichment of commutative R-algebras over sSet is given by:

$$X \otimes A = |A^{\wedge X_\bullet}|$$

for $X \in s\mathrm{Set}$ (the maps in the simplicial object in the right-hand side come from the unit and multiplication of A). Taking $X = S^1 = \Delta^1/\partial\Delta^1$ we get the Hochschild complex in the right hand side and the result follows.

4. Another construction of $THH(A)$

Another construction of THH can be given which is valid for algebras over the little intervals operad, D_1. We set the stage now for giving that construction.

Consider an operad P in Top. Then there is a symmetric monoidal topologically enriched category (over $(\mathrm{FinSet}, \amalg)$) associated to P, the so called category of operators of P or the May-Thomason construction of P (inspired by a similar construction by May and Thomason), which we denote $\underline{P}$. The object set of this category is $\mathbb{N}$ and the spaces of morphisms are:

$$\underline{P}(k, l) = \coprod_{f \in \mathrm{FinSet}(k,l)} \bigotimes_{i \in l} P(f^{-1}(i))$$

In particular, $\underline{P}(k,1) = P(k)$. The important property of this category is that a P-algebra A in a symmetric monoidal category C is the same as a symmetric monoidal functor $\underline{A} : \underline{P} \to C$.

Then the category $\underline{\mathsf{D}_1}$ can be easily described as:

$$\underline{\mathsf{D}_1}(k,l) = \{f \in \mathrm{Emb}([0,1]^{\amalg k}, [0,1]^{\amalg l}) : f \text{ preserves orientation, } f \text{ has locally constant speed}\}$$

and we can give a topological functor

$$\mathsf{D}[S^1] : \underline{D_1}^{\mathrm{op}} \to \mathrm{Top}$$

defined on objects by

$$\mathsf{D}[S^1](k) = \{f \in \mathrm{Emb}([0,1]^{\amalg k}, S^1) : f \text{ preserves orientation, } f \text{ has locally constant speed}\}$$

[Note that in the descriptions of $\underline{D_1}$ and $\mathsf{D}[S^1]$, the condition on the speed being locally constant does not change the homotopy type of the spaces.]

With this in place, we now have

$$THH(A) \simeq \mathsf{D}[S^1] \underset{\underline{\mathsf{D}_1}}{\otimes} \underline{A}$$

for an associative (cofibrant) R-algebra A. Note that the above expression makes sense for A a D_1-algebra in the category of R-modules. Also, it brings the oriented manifold S^1 explicitly into the picture.

5. Morita invariance

If A, B are (cofibrant) associative algebras and P is a A-B-bimodule (cofibrant as a R-module) then

$$THH(A, P \overset{\mathrm{L}}{\underset{B}{\otimes}} N) = THH(B, N \overset{\mathrm{L}}{\underset{A}{\otimes}} P) \tag{2}$$

for any A-B-bimodule N. Here, we put

$$M \overset{\mathrm{L}}{\underset{A}{\otimes}} N = |\,\mathrm{Bar}(M, A, N)|$$

Note that the map $M \otimes^{\mathrm{L}}_A N \to M \otimes_A N$ is a weak equivalence if M is a cofibrant right A-module. In particular, if P is a cofibrant A-B-bimodule, we get from (2) that

$$THH(A, P \underset{B}{\otimes} N) \simeq THH(B, N \underset{A}{\otimes} P)$$

From this, it follows that given a Morita equivalence between A and B, namely:

i. a A-B-bimodule P.
ii. a B-A-bimodule Q.
iii. a weak equivalence of A-bimodules $P \otimes^{\mathrm{L}}_B Q \simeq A$.
iv. a weak equivalence of B-bimodules $Q \otimes^{\mathrm{L}}_A P \simeq B$.

then

$$THH(A, M) \simeq THH(B, Q \overset{\mathrm{L}}{\underset{A}{\otimes}} M \overset{\mathrm{L}}{\underset{A}{\otimes}} P) \tag{3}$$

The proof is simple:

$$\begin{aligned}
THH(A,M) &\simeq THH(A, A \overset{\mathrm{L}}{\underset{A}{\otimes}} M) \\
&\simeq THH(A, P \overset{\mathrm{L}}{\underset{B}{\otimes}} Q \overset{\mathrm{L}}{\underset{A}{\otimes}} M) \\
&= THH(B, Q \overset{\mathrm{L}}{\underset{A}{\otimes}} M \overset{\mathrm{L}}{\underset{A}{\otimes}} P)
\end{aligned}$$

Actually, one can see from the proof that condition (iv) is not necessary. In view of a remark above, if P and Q are cofibrant (left and right, respectively) A-modules or M is a cofibrant A-bimodule then we can drop the superscripts “L” from (3).

The usual example of a Morita equivalence is between A and the matrix algebra $M_n(A) = \mathrm{Hom}_A(A^{\vee n}, A^{\vee n})$. Then the A-$M_n(A)$-bimodule $\mathrm{Hom}_A(A, A^{\vee n}) = A^{\vee n}$ and the $M_n(A)$-A-bimodule $\mathrm{Hom}_A(A^{\vee n}, A) = A^{\times n}$ ($\simeq A^{\vee n}$) give the desired Morita equivalence (where the weak equivalences in the conditions above actually become isomorphisms). This Morita equivalence translates to an equivalence

$$THH(A,M) \simeq THH(M_n(A), M_n(M))$$

as a particular case of (3).